Powers in a class of $\mathcal{A}$-strict standard episturmian words

Amy Glen

amy.glen@adelaide.edu.au

http://www.maths.adelaide.edu.au/∼aglen

The University of Adelaide

Introduction

- **Sturmian words**

Introduction

- **Sturmian words**
  - Infinite words with $n+1$ distinct factors of length n for each $n \in \mathbb{N}$.

Introduction

- **Sturmian words**
  - Infinite words with $n+1$ distinct factors of length n for each $n \in \mathbb{N}$.
  - Sturmian words are over a 2-letter alphabet.

Introduction

- **Sturmian words**
  - Infinite words with $n+1$ distinct factors of length n for each $n \in \mathbb{N}$.
  - Sturmian words are over a 2-letter alphabet.
  - They are exactly the aperiodic infinite words of minimal complexity.

Introduction

- **Sturmian words**
  - Infinite words with $n+1$ distinct factors of length n for each $n \in \mathbb{N}$.
  - Sturmian words are over a 2-letter alphabet.
  - They are exactly the aperiodic infinite words of minimal complexity.
- **Episturmian words**

Introduction

- **Sturmian words**
  - Infinite words with $n+1$ distinct factors of length n for each $n \in \mathbb{N}$.
  - Sturmian words are over a 2-letter alphabet.
  - They are exactly the aperiodic infinite words of minimal complexity.
- **Episturmian words**
  - A natural generalization of Sturmian words to an arbitrary finite alphabet.

Introduction

- **Sturmian words**
  - Infinite words with $n+1$ distinct factors of length n for each $n \in \mathbb{N}$.
  - Sturmian words are over a 2-letter alphabet.
  - They are exactly the aperiodic infinite words of minimal complexity.
- **Episturmian words**
  - A natural generalization of Sturmian words to an arbitrary finite alphabet.
  - Introduced by Droubay, Justin, and Pirillo (2001).

Introduction

- **Sturmian words**
  - Infinite words with $n+1$ distinct factors of length n for each $n \in \mathbb{N}$.
  - Sturmian words are over a 2-letter alphabet.
  - They are exactly the aperiodic infinite words of minimal complexity.
- **Episturmian words**
  - A natural generalization of Sturmian words to an arbitrary finite alphabet.
  - Introduced by Droubay, Justin, and Pirillo (2001).
  - Sturmian words are exactly the aperiodic episturmian words over a 2-letter alphabet.

Aim

- Explicitly determine all integer powers occurring in episturmian words.
- This has been done for Sturmian words by Damanik & Lenz (2003).
- We do this for a restricted class of episturmian words.

Terminology and notation

Let $\mathcal{A}$ denote a *finite alphabet* and let $u = x_1 x_2 \cdots x_m$ where each $x_i \in \mathcal{A}$.

Terminology and notation

Let $\mathcal{A}$ denote a *finite alphabet* and let $u = x_1 x_2 \cdots x_m$ where each $x_i \in \mathcal{A}$.

- The *length* of u is $|u| = m$.

Terminology and notation

Let $\mathcal{A}$ denote a *finite alphabet* and let $u = x_1x_2\cdots x_m$ where each $x_i \in \mathcal{A}$.

- The *length* of u is $|u| = m$.
- The *reversal* of u is $\widetilde{u} = x_m x_{m-1} \ldots x_1$.

Terminology and notation

Let $\mathcal{A}$ denote a *finite alphabet* and let $u = x_1 x_2 \cdots x_m$ where each $x_i \in \mathcal{A}$.

- The *length* of u is $|u| = m$.
- The *reversal* of u is $\widetilde{u} = x_m x_{m-1} \ldots x_1$.
- u is a *palindrome* if $u = \widetilde{u}$.

Terminology and notation

Let $\mathcal{A}$ denote a *finite alphabet* and let $u = x_1 x_2 \cdots x_m$ where each $x_i \in \mathcal{A}$.

- The *length* of u is $|u| = m$.
- The *reversal* of u is $\widetilde{u} = x_m x_{m-1} \ldots x_1$.
- u is a *palindrome* if $u = \widetilde{u}$.
- u^ω denotes the *purely periodic* infinite word $uuu \cdots$.

Terminology and notation

Let $\mathcal{A}$ denote a *finite alphabet* and let $u = x_1x_2\cdots x_m$ where each $x_i \in \mathcal{A}$.

- The *length* of u is $|u| = m$.
- The *reversal* of u is $\widetilde{u} = x_m x_{m-1} \ldots x_1$.
- u is a *palindrome* if $u = \widetilde{u}$.
- u^ω denotes the *purely periodic* infinite word $uuu\cdots$.
- For $0 \leq j \leq m-1$, the *j-th conjugate* of u is the word

$$C_j(u) := x_{j+1}x_{j+2}\cdots x_m x_1 x_2 \cdots x_j$$

and we define

$$\mathcal{C}(u) := \{C_j(u) \ : \ 0 \leq j \leq |u|-1\},$$

the *conjugacy class* of u.

Let $\mathbf{x}$ be an infinite word over $\mathcal{A}$.

- A *factor* of $\mathbf{x}$ is a finite string of consecutive letters in $\mathbf{x}$.

Let $\mathbf{x}$ be an infinite word over $\mathcal{A}$.

- A *factor* of $\mathbf{x}$ is a finite string of consecutive letters in $\mathbf{x}$.
- $\Omega(\mathbf{x})$ denotes the set of all factors of $\mathbf{x}$.
- $\Omega_n(\mathbf{x})$ denotes the set of all factors of $\mathbf{x}$ of length n.

Let $\mathbf{x}$ be an infinite word over $\mathcal{A}$.

- A *factor* of $\mathbf{x}$ is a finite string of consecutive letters in $\mathbf{x}$.
- $\Omega(\mathbf{x})$ denotes the set of all factors of $\mathbf{x}$.
- $\Omega_n(\mathbf{x})$ denotes the set of all factors of $\mathbf{x}$ of length n.
- If $w \in \Omega(\mathbf{x})$, we write $w \prec \mathbf{x}$.

Let $\mathbf{x}$ be an infinite word over $\mathcal{A}$.

- A *factor* of $\mathbf{x}$ is a finite string of consecutive letters in $\mathbf{x}$.
- $\Omega(\mathbf{x})$ denotes the set of all factors of $\mathbf{x}$.
- $\Omega_n(\mathbf{x})$ denotes the set of all factors of $\mathbf{x}$ of length n.
- If $w \in \Omega(\mathbf{x})$, we write $w \prec \mathbf{x}$.
- A factor w of $\mathbf{x}$ is

$$\begin{cases} \text{right} \\ \text{left} \end{cases} \quad \textit{special} \quad \text{if} \quad \begin{cases} wa, \ wb \\ aw, \ bw \end{cases}$$

are factors of $\mathbf{x}$ for some $a, b \in \mathcal{A}$, $a \neq b$.

Episturmian words

- An infinite word $\mathbf{t}$ is *episturmian* if:
  - $\Omega(\mathbf{t})$ is closed under reversal, and
  - $\mathbf{t}$ has at most one right special factor of each length.

Episturmian words

- An infinite word $\mathbf{t}$ is *episturmian* if:
  - $\Omega(\mathbf{t})$ is closed under reversal, and
  - $\mathbf{t}$ has at most one right special factor of each length.
- An episturmian word is *standard* if all of its left special factors are prefixes of it.

Standard episturmian words

- Let $\mathbf{t}$ be a standard episturmian word over $\mathcal{A}$ and let

$$u_1 = \varepsilon, \ u_2, \ u_3, \ u_4, \ \ldots$$

be the sequence of its palindromic prefixes.

Standard episturmian words

- Let $\mathbf{t}$ be a standard episturmian word over $\mathcal{A}$ and let

$$u_1 = \varepsilon, \ u_2, \ u_3, \ u_4, \ \ldots$$

be the sequence of its palindromic prefixes.

- $\exists$ an infinite word $\Delta(\mathbf{t}) = x_1 x_2 x_3 \cdots$ ($x_i \in \mathcal{A}$) such that

$$\boxed{u_{n+1} = (u_n x_n)^{(+)}, \quad n \in \mathbb{N}^+.}$$

Note: $w^{(+)}$ is the shortest palindrome of which w is a prefix.

Standard episturmian words

- Let $\mathbf{t}$ be a standard episturmian word over $\mathcal{A}$ and let

$$u_1 = \varepsilon, \ u_2, \ u_3, \ u_4, \ \ldots$$

be the sequence of its palindromic prefixes.

- $\exists$ an infinite word $\Delta(\mathbf{t}) = x_1x_2x_3\cdots$ ($x_i \in \mathcal{A}$) such that

$$\boxed{u_{n+1} = (u_nx_n)^{(+)}, \quad n \in \mathbb{N}^+.}$$

Note: $w^{(+)}$ is the shortest palindrome of which w is a prefix.

- $\Delta(\mathbf{t})$ is called the *directive word* of $\mathbf{t}$.

Standard episturmian words

- Let $\mathbf{t}$ be a standard episturmian word over $\mathcal{A}$ and let

$$u_1 = \varepsilon, \ u_2, \ u_3, \ u_4, \ \ldots$$

be the sequence of its palindromic prefixes.

- $\exists$ an infinite word $\Delta(\mathbf{t}) = x_1 x_2 x_3 \cdots$ ($x_i \in \mathcal{A}$) such that

$$\boxed{u_{n+1} = (u_n x_n)^{(+)}, \quad n \in \mathbb{N}^+.}$$

Note: $w^{(+)}$ is the shortest palindrome of which w is a prefix.

- $\Delta(\mathbf{t})$ is called the *directive word* of $\mathbf{t}$.
- $\mathbf{t} = \lim_{n\to\infty} u_n$

A class of strict standard episturmian words

Take $\mathcal{A} = \mathcal{A}_k := \{a_1, a_2, \ldots, a_k\}$.

A class of strict standard episturmian words

Take $\mathcal{A} = \mathcal{A}_k := \{a_1, a_2, \ldots, a_k\}$.

- $\mathbf{t}$ is *k-strict* if each $a_i \in \mathcal{A}_k$ appears infinitely often in $\Delta(\mathbf{t})$.

A class of strict standard episturmian words

Take $\mathcal{A} = \mathcal{A}_k := \{a_1, a_2, \ldots, a_k\}$.

- $\mathbf{t}$ is *k-strict* if each $a_i \in \mathcal{A}_k$ appears infinitely often in $\Delta(\mathbf{t})$.
- k-strict episturmian words are precisely the k-letter Arnoux-Rauzy sequences.

A class of strict standard episturmian words

Take $\mathcal{A} = \mathcal{A}_k := \{a_1, a_2, \ldots, a_k\}$.

- $\mathbf{t}$ is *k-strict* if each $a_i \in \mathcal{A}_k$ appears infinitely often in $\Delta(\mathbf{t})$.
- k-strict episturmian words are precisely the k-letter Arnoux-Rauzy sequences.
- For any standard episturmian word $\mathbf{t}$,

$$\Delta(\mathbf{t}) = a_1^{d_1} a_2^{d_2} \cdots a_k^{d_k} a_1^{d_{k+1}} a_2^{d_{k+2}} \cdots a_k^{d_{2k}} a_1^{d_{2k+1}} \cdots ,$$

  where each $d_i \geq 0$.

A class of strict standard episturmian words

Take $\mathcal{A} = \mathcal{A}_k := \{a_1, a_2, \ldots, a_k\}$.

- $\mathbf{t}$ is *k-strict* if each $a_i \in \mathcal{A}_k$ appears infinitely often in $\Delta(\mathbf{t})$.
- k-strict episturmian words are precisely the k-letter Arnoux-Rauzy sequences.
- For any standard episturmian word $\mathbf{t}$,

$$\Delta(\mathbf{t}) = a_1^{d_1} a_2^{d_2} \cdots a_k^{d_k} a_1^{d_{k+1}} a_2^{d_{k+2}} \cdots a_k^{d_{2k}} a_1^{d_{2k+1}} \cdots ,$$

  where each $d_i \geq 0$.
- We restrict our attention to the case when all $d_i > 0$.
- Let $\mathbf{s}$ be the k-strict standard episturmian word with directive word:

$$\Delta(\mathbf{s}) = a_1^{d_1} a_2^{d_2} \cdots a_k^{d_k} a_1^{d_{k+1}} a_2^{d_{k+2}} \cdots a_k^{d_{2k}} a_1^{d_{2k+1}} \cdots , \quad d_i > 0.$$

Example

- Let $\alpha \in (0,1)$ be irrational with $\alpha = [0; 1 + d_1, d_2, d_3, \ldots]$.
- The *characteristic Sturmian word* c_α over $\{a, b\}$ has directive word

$$\Delta(c_\alpha) = a^{d_1} b^{d_2} a^{d_3} b^{d_4} a^{d_5} \cdots .$$

Example

- Let $\alpha \in (0,1)$ be irrational with $\alpha = [0; 1+d_1, d_2, d_3, \ldots]$.
- The *characteristic Sturmian word* c_α over $\{a,b\}$ has directive word

$$\Delta(c_\alpha) = a^{d_1} b^{d_2} a^{d_3} b^{d_4} a^{d_5} \cdots .$$

- $c_\alpha = \lim_{n\to\infty} s_n$, where $(s_n)_{n\geq -1}$ is defined by

$$s_{-1} = b, \quad s_0 = a, \quad s_n = s_{n-1}^{d_n} s_{n-2}, \quad n \geq 1.$$

Example

- Let $\alpha \in (0,1)$ be irrational with $\alpha = [0; 1+d_1, d_2, d_3, \ldots]$.
- The *characteristic Sturmian word* c_α over $\{a, b\}$ has directive word

$$\Delta(c_\alpha) = a^{d_1} b^{d_2} a^{d_3} b^{d_4} a^{d_5} \cdots .$$

- $c_\alpha = \lim_{n\to\infty} s_n$, where $(s_n)_{n\geq -1}$ is defined by

$$s_{-1} = b, \quad s_0 = a, \quad s_n = s_{n-1}^{d_n} s_{n-2}, \quad n \geq 1.$$

- $\Delta(\mathbf{s})$ resembles $\Delta(c_\alpha)$.

Example

- Let $\alpha \in (0,1)$ be irrational with $\alpha = [0; 1+d_1, d_2, d_3, \ldots]$.
- The *characteristic Sturmian word* c_α over $\{a,b\}$ has directive word

$$\Delta(c_\alpha) = a^{d_1} b^{d_2} a^{d_3} b^{d_4} a^{d_5} \cdots .$$

- $c_\alpha = \lim_{n\to\infty} s_n$, where $(s_n)_{n\geq -1}$ is defined by

$$s_{-1} = b, \quad s_0 = a, \quad s_n = s_{n-1}^{d_n} s_{n-2}, \quad n \geq 1.$$

- $\Delta(\mathbf{s})$ resembles $\Delta(c_\alpha)$.
- One can prove that $\mathbf{s} = \lim_{n\to\infty} s_n$ where the sequence $(s_n)_{n\geq 1-k}$ is defined by

$$s_{1-k} = a_2, \quad s_{2-k} = a_3, \quad \ldots, \quad s_{-1} = a_k, \quad s_0 = a_1,$$

$$s_n = s_{n-1}^{d_n} s_{n-2}^{d_{n-1}} \cdots s_0^{d_1} a_{n+1}, \quad 1 \leq n \leq k-1,$$

$$s_n = s_{n-1}^{d_n} s_{n-2}^{d_{n-1}} \cdots s_{n-k+1}^{d_{n-k+2}} s_{n-k}, \quad n \geq k.$$

Powers

- Let $p \geq 2$ be an integer.
- A finite word w has a *p-th power* in $\mathbf{s}$ if

$$w^p = \underbrace{ww \cdots w}_{p} \quad \text{is a factor of } \mathbf{s}.$$

Powers

- Let $p \geq 2$ be an integer.
- A finite word w has a *p-th power* in $\mathbf{s}$ if

$$w^p = \underbrace{ww \cdots w}_{p} \quad \text{is a factor of } \mathbf{s}.$$

Key tools in our analysis of powers occurring in $\mathbf{s}$:

- canonical decompositions of $\mathbf{s}$ in terms of its building blocks s_n;
- a generalization of singular words.

Singular words

- The set of factors of length $|s_n|$ in c_α is given by

$$\{\text{all conjugates of } s_n\} \cup \{w_n\}$$

where w_n is called the n-th *singular factor* of c_α.

[Wen and Wen (1994), Melançon (1999), Cao and Wen (2003)]

Singular words

- The set of factors of length $|s_n|$ in c_α is given by

$$\{\text{all conjugates of } s_n\} \cup \{w_n\}$$

where w_n is called the n-th *singular factor* of c_α.

[Wen and Wen (1994), Melançon (1999), Cao and Wen (2003)]

Singular n-words of the i-th kind

- The set of factors of $\mathbf{s}$ of length $|s_n|$ can be partitioned into k sets.

Singular words

- The set of factors of length $|s_n|$ in c_α is given by

$$\{\text{all conjugates of } s_n\} \cup \{w_n\}$$

where w_n is called the n-th *singular factor* of c_α.

[Wen and Wen (1994), Melançon (1999), Cao and Wen (2003)]

Singular n-words of the i-th kind

- The set of factors of s of length $|s_n|$ can be partitioned into k sets.
- That is:

$$\boxed{\Omega_{|s_n|}(\mathbf{s}) = \mathcal{C}(s_n)\,\dot{\cup}\,\Omega_n^1\,\dot{\cup}\cdots\dot{\cup}\,\Omega_n^{k-1}.}$$

THE UNIVERSITY OF ADELAIDE AUSTRALIA

Singular words

- The set of factors of length $|s_n|$ in c_α is given by

$$\{\text{all conjugates of } s_n\} \cup \{w_n\}$$

where w_n is called the n-th *singular factor* of c_α.

[Wen and Wen (1994), Melançon (1999), Cao and Wen (2003)]

Singular n-words of the i-th kind

- The set of factors of $\mathbf{s}$ of length $|s_n|$ can be partitioned into k sets.
- That is:

$$\boxed{\Omega_{|s_n|}(\mathbf{s}) = \mathcal{C}(s_n) \,\dot\cup\, \Omega_n^1 \,\dot\cup \cdots \dot\cup\, \Omega_n^{k-1}.}$$

- Each set Ω_n^i is closed under reversal.

Singular words

- The set of factors of length $|s_n|$ in c_α is given by

$$\{\text{all conjugates of } s_n\} \cup \{w_n\}$$

  where w_n is called the n-th *singular factor* of c_α.

  [Wen and Wen (1994), Melançon (1999), Cao and Wen (2003)]

Singular n-words of the i-th kind

- The set of factors of $\mathbf{s}$ of length $|s_n|$ can be partitioned into k sets.
- That is:

$$\boxed{\Omega_{|s_n|}(\mathbf{s}) = \mathcal{C}(s_n) \,\dot\cup\, \Omega_n^1 \,\dot\cup \cdots \dot\cup\, \Omega_n^{k-1}.}$$

- Each set Ω_n^i is closed under reversal.
- If $w \in \Omega_n^i$ then w is called a *singular n-word of the i-th kind.*
- Such words play a key role in our study of powers occurring in $\mathbf{s}$.

Powers occurring in s

- Let $n \in \mathbb{N}^+$ be fixed.
- We define k sets of lengths between $|s_n|$ and $|s_{n+1}|$:

$$\mathcal{D}_1(n) := \{r|s_n| \ : \ 1 \leq r \leq d_{n+1}\},$$

$$\mathcal{D}_i(n) := \{|s_n^r s_{n-1}^{d_n} \cdots s_{n+2-i}^{d_{n+3-i}} s_{n+1-i}| \ : \ 1 \leq r \leq d_{n+1}\}, \quad 2 \leq i \leq k-1,$$

$$\mathcal{D}_k(n) := \{|s_n^r s_{n-1}^{d_n} \cdots s_{n+2-k}^{d_{n+3-k}} s_{n+1-k}| \ : \ 1 \leq r \leq d_{n+1} - 1\}.$$

Powers occurring in $\mathbf{s}$

- Let $n \in \mathbb{N}^+$ be fixed.
- We define k sets of lengths between $|s_n|$ and $|s_{n+1}|$:

$$\mathcal{D}_1(n) := \{r|s_n| \ : \ 1 \leq r \leq d_{n+1}\},$$

$$\mathcal{D}_i(n) := \{|s_n^r s_{n-1}^{d_n} \cdots s_{n+2-i}^{d_{n+3-i}} s_{n+1-i}| \ : \ 1 \leq r \leq d_{n+1}\}, \quad 2 \leq i \leq k-1,$$

$$\mathcal{D}_k(n) := \{|s_n^r s_{n-1}^{d_n} \cdots s_{n+2-k}^{d_{n+3-k}} s_{n+1-k}| \ : \ 1 \leq r \leq d_{n+1} - 1\}.$$

Let $\mathcal{D}_n = \bigcup_{i=1}^k \mathcal{D}_i(n)$.

Powers occurring in $\mathbf{s}$

- Let $n \in \mathbb{N}^+$ be fixed.
- We define k sets of lengths between $|s_n|$ and $|s_{n+1}|$:

$$\mathcal{D}_1(n) := \{r|s_n| \ : \ 1 \leq r \leq d_{n+1}\},$$

$$\mathcal{D}_i(n) := \{|s_n^r s_{n-1}^{d_n} \cdots s_{n+2-i}^{d_{n+3-i}} s_{n+1-i}| \ : \ 1 \leq r \leq d_{n+1}\}, \quad 2 \leq i \leq k-1,$$

$$\mathcal{D}_k(n) := \{|s_n^r s_{n-1}^{d_n} \cdots s_{n+2-k}^{d_{n+3-k}} s_{n+1-k}| \ : \ 1 \leq r \leq d_{n+1} - 1\}.$$

Let $\mathcal{D}_n = \bigcup_{i=1}^{k} \mathcal{D}_i(n)$.

- Suppose $w \prec \mathbf{s}$ and let $p \geq 2$ be an integer. Then,

$$\boxed{w^p \prec \mathbf{s} \quad \Rightarrow \quad |w| \in \mathcal{D}_n \text{ for some } n.}$$

THE UNIVERSITY OF ADELAIDE AUSTRALIA

Squares

- In $\mathbf{s}$, successive occurrences of a singular word are *positively separated.*

Squares

- In $\mathbf{s}$, successive occurrences of a singular word are *positively separated.*

- Consequently:

  Lemma: *Suppose* $u^2 \prec \mathbf{s}$ *with* $|u| \in \mathcal{D}_n$. *Then*

  $$w \not\prec u \quad \textit{if} \quad w \in \Omega^1_{n+1-i} \textit{ for some } i \in [1, k-1].$$

Squares

- In $\mathbf{s}$, successive occurrences of a singular word are *positively separated.*
- Consequently:

  Lemma: *Suppose* $u^2 \prec \mathbf{s}$ *with* $|u| \in \mathcal{D}_n$. *Then*

  $$w \not\prec u \quad \textit{if} \quad w \in \Omega^1_{n+1-i} \textit{ for some } i \in [1, k-1].$$

  - That is, u does not contain a singular $(n+1-i)$-word of the first kind for any $i \in [1, k-1]$.

THE UNIVERSITY OF ADELAIDE AUSTRALIA

Squares, cubes, and higher powers

- Let $w \prec \mathrm{s}$ with $|w| \in \mathcal{D}_n$ for some n.

THE UNIVERSITY OF ADELAIDE AUSTRALIA

Squares, cubes, and higher powers

- Let $w \prec \mathrm{s}$ with $|w| \in \mathcal{D}_n$ for some n.
- Our main results show:

  If $w^p \prec \mathrm{s}$, then w is a conjugate of a finite product of blocks from the set $\{s_n, s_{n-1}, \ldots, s_{n+1-k}\}$, depending on $|w|$ and d_{n+1}.

Squares, cubes, and higher powers

- Let $w \prec \mathrm{s}$ with $|w| \in \mathcal{D}_n$ for some n.
- Our main results show:

  If $w^p \prec \mathrm{s}$, then w is a conjugate of a finite product of blocks from the set $\{s_n, s_{n-1}, \ldots, s_{n+1-k}\}$, depending on $|w|$ and d_{n+1}.
- For instance:
  - Let $p \geq 2$.
  - Suppose $|w| = r|s_n|$ for some r with $1 \leq r < (d_{n+1} + 2)/p$.

Squares, cubes, and higher powers

- Let $w \prec \mathrm{s}$ with $|w| \in \mathcal{D}_n$ for some n.
- Our main results show:

  If $w^p \prec \mathrm{s}$, then w is a conjugate of a finite product of blocks from the set $\{s_n, s_{n-1}, \ldots, s_{n+1-k}\}$, depending on $|w|$ and d_{n+1}.
- For instance:
  - Let $p \geq 2$.
  - Suppose $|w| = r|s_n|$ for some r with $1 \leq r < (d_{n+1} + 2)/p$.
  - Then:

$$w^p \prec \mathrm{s} \quad \Longleftrightarrow \quad w \text{ is one of the first } |s_n| \text{ conjugates of } (s_n)^r.$$

Example: k-bonacci word

- Define the *k-bonacci word* to be the standard episturmian word $\boldsymbol{\eta}_k$ with directive word $(a_1 a_2 \cdots a_k)^\omega$.

Example: k-bonacci word

- Define the *k-bonacci word* to be the standard episturmian word $\boldsymbol{\eta}_k$ with directive word $(a_1 a_2 \cdots a_k)^\omega$.
- Since all $d_i = 1$, we have $s_n = s_{n-1} s_{n-2} \cdots s_{n-k}$ for all $n \geq 1$.

  (The lengths $|s_n|$ are the *k-bonacci numbers*.)

Example: k-bonacci word

- Define the *k-bonacci word* to be the standard episturmian word $\boldsymbol{\eta}_k$ with directive word $(a_1 a_2 \cdots a_k)^\omega$.
- Since all $d_i = 1$, we have $s_n = s_{n-1} s_{n-2} \cdots s_{n-k}$ for all $n \geq 1$.

  (The lengths $|s_n|$ are the *k-bonacci numbers*.)
- If $w^p \prec \boldsymbol{\eta}_k$, then

  $|w| = |s_n| + |s_{n-1}| + \cdots + |s_{n+1-i}|$ for some $n \in \mathbb{N}$ and $i \in [1, k-1]$.

THE UNIVERSITY OF ADELAIDE AUSTRALIA

- Our main results reveal that, in $\boldsymbol{\eta}_k$,
  - $(a_1)^2$ is the unique square of length 2;
  - all conjugates of s_n have a square;
  - only some conjugates of s_n have a cube;
  - only some conjugates of $s_n s_{n-1} \cdots s_{n+1-i}$ have a square.

- Our main results reveal that, in $\boldsymbol{\eta}_k$,
  - $(a_1)^2$ is the unique square of length 2;
  - all conjugates of s_n have a square;
  - only some conjugates of s_n have a cube;
  - only some conjugates of $s_n s_{n-1} \cdots s_{n+1-i}$ have a square.
- There are no other integer powers in $\boldsymbol{\eta}_k$.
  - In particular, the k-bonacci word is 4-power free.

Concluding remarks

- Our main results on powers suffice to describe all integer powers occurring in any (episturmian) word that is equivalent to $\mathbf{s}$.
- **Open problem:**

  Determine all integer powers occurring in general standard episturmian words (with not all d_i necessarily positive).